

HARNACK INEQUALITY AND REGULARITY FOR DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We prove Harnack inequality and local regularity results for weak solutions of a quasilinear degenerate equation in divergence form under natural growth conditions. The degeneracy is given by a suitable power of a strong A_∞ weight. Regularity results are achieved under minimal assumptions on the coefficients and, as an application, we prove $C^{1,\alpha}$ local estimates for solutions of a degenerate equation in non divergence form.

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1. INTRODUCTION

This paper contains a contribution towards a complete theory concerning regularity and smoothness for solutions of degenerate elliptic equations under minimal assumptions on the coefficients. Here we consider quasilinear elliptic equations whose ellipticity degenerates as a suitable power of a strong A_∞ weight. The class of strong A_∞ weights has been introduced by David and Semmes in [2] and it is useful in several problems related to geometric measure theory and quasiconformal mappings. Indeed, as it is well known, the Jacobian of a quasiconformal mapping is a strong A_∞ weight. Weights of this kind enjoy some metric properties and important inequalities like Poincarè and Sobolev's. Moreover any strong A_∞ weight is a Muckenhoupt weight, and there exist A_2 weights which are not strong A_∞ weights. The only Muckenhoupt weights whose degeneration gives

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regularity for elliptic equations are those in the class A_2 (see e.g. [6], [12], [19], [20], [4]).

Let us consider quasilinear elliptic equations in divergence form

$$(1) \quad \operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0,$$

where A and B are measurable functions satisfying suitable structure conditions

$$(2) \quad \begin{cases} |A(x, u, \xi)| \leq a\omega(x)|\xi|^{p-1} + b(x)|u|^{p-1} + e(x) \\ |B(x, u, \xi)| \leq b_0\omega(x)|\xi|^p + b_1(x)|\xi|^{p-1} + d(x)|u|^{p-1} + f(x) \\ \xi \cdot A(x, u, \xi) \geq \omega|\xi|^p - d(x)|u|^p - g(x). \end{cases}$$

Here $1 < p < n$, $\omega = v^{1-\frac{p}{n}}$, v is a strong A_∞ weight and the coefficients of the lower order terms belong to suitable Stummel - Kato or Morrey classes. The function B is required to have natural growth in the variable ξ .

Equation (1) with $v \equiv 1$ has extensively been investigated. Here we quote some contributions - among others - by Trudinger and Lieberman. In [18] (see also [11]) Trudinger considers the same equation with no degeneracy and coefficients in suitable L^p classes. There, Harnack inequality and regularity properties of bounded weak solutions are proved. In [14] Lieberman considers equation

$$(3) \quad \operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = \mu$$

assuming μ to be a given signed Radon measure satisfying a Morrey type condition. There, Harnack inequality and regularity for bounded weak solutions are proved under the structure conditions (2) with lower order terms in suitable Morrey classes.

Our results are parallel to those in [18] and [14]. We follow the pattern drawn in [18].

The novelty here is the special kind of degeneracy. We assume the coefficients in suitable Stummel-Kato and Morrey classes and - as a technique - use a Fefferman type inequality proved in [5] to control the integrals arising from the lower order terms. The inequality is based on a representation formula proved in [7] (see also [8]).

In Section 4, as an application of the previous results, we prove $C^{1,\alpha}$ estimates for a non variational elliptic equation related to equation (1).

2. STRONG A_∞ WEIGHTS AND FUNCTION SPACES

Let v be an A_∞ weight in \mathbb{R}^n . This means that, for any $\varepsilon > 0$ there exists $\delta > 0$, such that if Q is a cube in \mathbb{R}^n and E is a measurable subset of Q for which $|E| \leq \delta|Q|$ holds, then $v(E) \leq \varepsilon v(Q)$, i.e. $\int_E v(x) dx \leq \varepsilon \int_Q v(x) dx$. If $v \in A_\infty$ and $B_{x,y}$ is the euclidean ball containing x and y

with diameter $|x - y|$, we can define a quasi distance δ in \mathbb{R}^n by setting

$$\delta(x, y) = \left(\int_{B_{x,y}} v(t) dt \right)^{1/n}.$$

We remark that $\delta(x, y) = |x - y|$ when $v(t) \equiv 1$. By using the function $\delta(x, y)$ we may define the δ -length of a curve as the limsup of the δ -lengths of the approximating polygonals.

On the other side we can actually define a distance related to the weight v . We take, as the distance between two points x and y , the infimum of the δ -length of the curves connecting x and y . Namely we set,

$$d_v(x, y) = \inf \{ \delta\text{-length of the curves connecting } x \text{ and } y \}.$$

In general, the function δ is not comparable to a distance.

Definition 2.1. *If v is an A_∞ weight there exists a positive constant c such that $\delta(x, y) \leq c d_v(x, y)$, for any $x, y \in \mathbb{R}^n$ (see [2]). If, in addition,*

$$(4) \quad \delta(x, y) \sim d_v(x, y) \quad \forall x, y \in \mathbb{R}^n$$

we say that v is a strong A_∞ weight.

The measure $v dx$ is Ahlfors regular and, as a consequence, is a doubling measure (see e.g. [16]).

In this section we denote by $B \equiv B(x, R)$ and $B_e \equiv B(x, R)$ respectively the metric and euclidean balls centered at x with radius R .

Theorem 2.1. *Let v be a strong A_∞ weight. Then, there exist two positive constants a and A , depending only on n and the comparability constants in (4), such that for any $x \in \mathbb{R}^n$ and any $r > 0$, we have*

$$a r^n \leq v(B(x, r)) \leq A r^n.$$

Moreover, there exists $c > 0$ such that for any $r > 0$ there exists $R = R(r)$ such that

$$B_e(x, cR) \subseteq B(x, r) \subseteq B_e(x, R) \quad \forall x \in \mathbb{R}^n.$$

It is possible to compare the A_p classes of Muckenhoupt weights and strong A_∞ .

Remark 2.1. *Any strong A_∞ weight is a A_∞ weight. For any $1 < p < \infty$ there exists an A_p weight which is not a strong A_∞ weight.*

In [2] David and Semmes show that Poincaré and Sobolev inequalities hold true for strong A_∞ weights. Unfortunately, they prove inequalities with different weights on both sides and, in order to run Moser iteration, we need Poincaré and Sobolev inequalities with the same weight on both sides. However, in [13] Sobolev and Poincaré inequalities are proved as a consequence of the results in [2] and [7], and in [9] it is shown how to pass from a Poincaré inequality with two different weights to a Poincaré inequality with the same weight on both sides.

We quote only the results in the form we need. Our statement can be easily derived from the above cited papers.

Theorem 2.2 ([13]). *Let v be a strong A_∞ weight and $1 < p < n$. Let q be such that $v \in A_q$. The following Sobolev inequality and Poincaré inequality hold true*

$$(5) \quad \left(\int_B |u(x)|^{kp} \omega \, dx \right)^{\frac{1}{kp}} \leq c \, \text{diam}(B) \left(\int_B |\nabla u(x)|^p \omega \, dx \right)^{\frac{1}{p}} \quad \forall u \in C_0^\infty(B)$$

$$(6) \quad \int_B |u - u_B|^p \omega \, dx \leq c(\text{diam } B)^p \int_B |\nabla u|^p \omega \, dx \quad \forall u \in C^\infty(B)$$

where $\omega(x) = v(x)^{1-\frac{p}{n}}$, and \int_B denotes the average with respect to the measure $\omega(x) \, dx$, $k = \frac{p+q(n-p)}{q(n-p)}$ and B denotes a metric ball.

Using strong A_∞ weights we define Lebesgue and Sobolev classes.

Definition 2.2. *Let v be a strong A_∞ weight and $\omega = v^{1-p/n}$, $\Omega \subset \mathbb{R}^n$. For any $u \in C_0^\infty(\Omega)$ we set*

$$(7) \quad \|u\|_{p,v} = \left(\int_\Omega |u(x)|^p \omega(x) \, dx \right)^{1/p} \quad 1 \leq p < \infty.$$

We define $L_v^p(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the above norm. In a similar way we define Sobolev classes. Let $1 < p < n$. For any $u \in C^\infty(\Omega)$ we set

$$(8) \quad \|u\|_{1,p,v} = \left(\int_\Omega |u(x)|^p \omega(x) \, dx \right)^{1/p} + \left(\int_\Omega |\nabla u(x)|^p \omega(x) \, dx \right)^{1/p}.$$

We define $H_{0,v}^{1,p}(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the above norm and $H_v^{1,p}(\Omega)$ to be the completion of $C^\infty(\Omega)$ with respect to the same norm.

Remark 2.2. In the above definitions we put v in the symbol of the norm and ω into the integrals. This is because we want to stress the dependence on the strong A_∞ weight v .

Remark 2.3. In general, i.e. if $\omega \notin A_2$, the classes H and W are different. Here we are going to study regularity of weak solutions taking H as a class of test functions (see [10] and [17]).

In order to formulate the assumptions on the lower order terms we need to define some other function spaces.

Definition 2.3. Let f be a locally integrable function in $\Omega \subset \mathbb{R}^n$ and let v be a strong A_∞ weight. We set

(9)

$$\phi(f; R) = \sup_{x \in \Omega} \left(\int_{B(x, R)} k(x, y) \left(\int_{B(x, R)} |f(z)| k(z, y) v(z)^{1-\frac{p}{n}} dz \right)^{\frac{1}{p-1}} v(y) dy \right)^{p-1}$$

where

$$k(x, y) = \frac{1}{v(B(x, d_v(x, y)))^{1-\frac{1}{n}}}.$$

We shall say that f belongs to the class $\tilde{S}_v(\Omega)$ if $\phi(f; R)$ is bounded in a neighborhood of the origin. Moreover, if $\lim_{R \rightarrow 0} \phi(f; R) = 0$ then we say that f belongs to the Stummel-Kato class $S_v(\Omega)$. If there exists $\rho > 0$ such that

$$(10) \quad \int_0^\rho \frac{\phi(f; t)^{1/p}}{t} dt < +\infty,$$

then we say that the function f belongs to the class $S'_v(\Omega)$.

Definition 2.4 (Morrey spaces). Let $p \in [1, +\infty[$ and v be a strong A_∞ weight. We say that f belongs to $L_v^{p, \lambda}(\Omega)$, for some $\lambda > 0$, if

$$\begin{aligned} \|f\|_{L_v^{p, \lambda}(\Omega)} &= \\ &= \sup_{x \in \Omega, 0 < r < d_0} \left(\frac{r^\lambda}{|B(x, r) \cap \Omega|} \int_{B(x, r) \cap \Omega} |f(y)|^p v(y)^{1-\frac{p}{n}} dy \right)^{\frac{1}{p}} < +\infty, \end{aligned}$$

where $d_0 = \text{diam}(\Omega)$.

Remark 2.4. It is an easy task to check that the above definitions give back their classical counterparts when $v \equiv 1$.

It is easy to compare the function classes previously defined.

Proposition 2.1. *Let p and ε be numbers such that $1 < p < n$ and $0 < \varepsilon < p$. The class $L_v^{1,p-\varepsilon}(\Omega)$ is embedded in $S'_v(\Omega)$. Namely, if $V \in L_v^{1,p-\varepsilon}(\Omega)$ then*

$$\phi(V; r) \leq C \|V\|_{L_v^{1,p-\varepsilon}} r^{\frac{\varepsilon}{p-1}}$$

for any $0 < r < \text{diam } \Omega$.

Proof. The proof is standard and can be easily adapted from the case $\omega = 1$ (see [3]). \square

The following two lemmas will be useful in the iteration process.

Lemma 2.1 ([15]). *Let $\mu(r)$ a continuous positive increasing function defined in $]0, +\infty[$ such that $\lim_{r \rightarrow 0} \mu(r) = 0$, $0 < \theta < 1$. The series*

$$\sum_{j=0}^{+\infty} \theta^j \log \mu^{-1}(\theta^{2j})$$

is convergent if and only if there exists $\rho > 0$ such that condition (10) is satisfied.

Lemma 2.2 ([15]). *Let $0 < \gamma < 1$, $h :]0, +\infty[\rightarrow]0, +\infty[$ a non decreasing function with $\lim_{t \rightarrow 0} h(t) = 0$, such that*

$$h(t) \leq Ch(t/2) \quad (C > 1)$$

and $\omega :]0, +\infty[\rightarrow]0, +\infty[$ a non decreasing function.

If

$$\omega(\rho) \leq \gamma\omega(4\rho) + h(\rho) \quad \forall \rho < \rho_0 < 1$$

then there exist $\bar{\rho} \leq \rho_0$, $0 < \sigma \leq 1$ and a positive constant K such that

$$\omega(\rho) \leq Kh^\sigma(\rho) \quad \forall \rho < \bar{\rho}.$$

The following result will be useful in the proof of the weak Harnack inequality.

Theorem 2.3 ([5]). *Let Ω be a bounded domain in \mathbb{R}^n and let V belong to the class $\tilde{S}_v(\Omega)$. If v is a strong A_∞ weight and $1 < p < n$, then there exists*

a constant c such that for any $u \in C_0^\infty(\Omega)$ we have

$$(11) \quad \left(\int_B |V(x)| |u(x)|^p \omega \, dx \right)^{1/p} \leq c \phi^{1/p}(V; 2R) \left(\int_B |\nabla u(x)|^p \omega \, dx \right)^{1/p}$$

where $\omega(x) \equiv v^{1-\frac{p}{n}}(x)$ and R is the radius of a metric ball $B \equiv B_R$, containing the support of u .

As a direct consequence we have

Corollary 2.1. *Let $1 < p < n$ and v be a strong A_∞ weight. Let V belongs to the class $S_v(\Omega)$. Then, for any $\varepsilon > 0$, there exists $K(\varepsilon)$ such that*

$$(12) \quad \int_\Omega |V(x)| |u(x)|^p \omega(x) \, dx \leq \varepsilon \int_\Omega |\nabla u(x)|^p \omega(x) \, dx + \\ + K(\varepsilon) \int_\Omega |u(x)|^p \omega(x) \, dx \quad \forall u \in C_0^\infty(\Omega)$$

where $\omega(x) = v(x)^{1-\frac{p}{n}}$, $K(\varepsilon) \sim \frac{\sigma}{[\phi^{-1}(V; \varepsilon)]^{n+p}}$ and ϕ^{-1} denotes the inverse function of ϕ .

3. HARNACK INEQUALITY

In this section we shall prove a weak Harnack inequality for non negative weak solutions of the equation

$$(13) \quad \operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0.$$

We recall what we mean by weak solution of (13).

Definition 3.1. *A function $u \in H_v^{1,p}(\Omega)$ is a local weak subsolution (super-solution) of equation (13) in Ω if*

$$(14) \quad \int_\Omega A(x, u(x), \nabla u(x)) \cdot \nabla \varphi \, dx - \int_\Omega B(x, u(x), \nabla u(x)) \varphi \, dx \leq 0 \quad (\geq 0)$$

for every $\varphi \in H_{0,v}^{1,p}(\Omega)$. A function u is a weak solution if it is both super and sub solution.

We require the functions $A(x, u, p)$ and $B(x, u, p)$ to be measurable functions satisfying the following structure conditions

$$(15) \quad \begin{cases} |A(x, u, \xi)| \leq a\omega(x)|\xi|^{p-1} + b(x)|u|^{p-1} + e(x) \\ |B(x, u, \xi)| \leq b_0\omega(x)|\xi|^p + b_1(x)|\xi|^{p-1} + d(x)|u|^{p-1} + f(x) \\ \xi \cdot A(x, u, \xi) \geq \omega|\xi|^p - d(x)|u|^p - g(x) \end{cases}$$

where $1 < p < n$, $\omega = v^{1-\frac{p}{n}}$ and v is a strong A_∞ weight.

We shall show that locally bounded weak solutions verify a Harnack inequality and, as a consequence, some regularity properties. We shall make the following assumptions on the lower order terms to ensure the continuity of local weak solutions

$$(16) \quad a, b_0 \in \mathbb{R}, \left(\frac{b}{\omega}\right)^{\frac{p}{p-1}}, \left(\frac{b_1}{\omega}\right)^p, \frac{d}{\omega}, \left(\frac{e}{\omega}\right)^{\frac{p}{p-1}}, \frac{f}{\omega}, \frac{g}{\omega} \in S'_v(\Omega).$$

From now on we denote by $B_r = B_r(x)$ the euclidean ball centered at x with radius r .

Theorem 3.1. *Let u be a non negative weak supersolution of equation (13) in Ω satisfying (15) and (16). Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \leq M$ in B_{3r} . Then there exists c depending on n, M, a_0, b_0, p and the weight v such that*

$$\omega^{-1}(B_{2r}) \int_{B_{2r}} u \omega dx \leq c \{ \min_{B_r} u + h(r) \}$$

$$\text{where } h(r) = \left[\phi \left(\left(\frac{e}{\omega} \right)^{\frac{p}{p-1}}; 3r \right) + \phi \left(\frac{g}{\omega}; 3r \right) \right]^{\frac{1}{p}} + \left[\phi \left(\frac{f}{\omega}; 3r \right) \right]^{\frac{1}{p-1}}.$$

Proof. We simplify the structure assumptions by setting $w = u + h(r)$. We get

$$(17) \quad \begin{cases} |A(x, u, \xi)| \leq a\omega(x)|\xi|^{p-1} + b_2(x)|w|^{p-1} \\ |B(x, u, \xi)| \leq b_0\omega(x)|\xi|^p + b_1(x)|\xi|^{p-1} + d_1(x)|w|^{p-1} \\ \xi \cdot A(x, u, \xi) \geq \omega(x)|\xi|^p - d_1(x)|w|^p \end{cases}$$

where $b_2 = b + h^{1-p}e$ and $d_1 = d + h^{1-p}f + h^{-p}g$. It is easy to check that b_2 and d_1 verify the same assumptions of b and d .

We take $\varphi = \eta^p w^\beta e^{-b_0 w}$, $\beta < 0$ as test function in (14) so we obtain

$$\begin{aligned} & \int_{B_{3r}} \eta^p e^{-b_0 w} (b_0 w^\beta + |\beta| w^{\beta-1}) \nabla w \cdot A - \\ & p \int_{B_{3r}} w^\beta \eta^{p-1} e^{-b_0 w} \nabla \eta \cdot A + \int_{B_{3r}} \eta^p w^\beta e^{-b_0 w} B \leq 0. \end{aligned}$$

The previous inequality and the structure assumptions (17) yield

$$\begin{aligned}
& \int_{B_{3r}} e^{-b_0 w} \eta^p (b_0 w^\beta + |\beta| w^{\beta-1}) |\nabla w|^p \omega dx \leq \\
& \int_{B_{3r}} e^{-b_0 w} \eta^p (b_0 w^\beta + |\beta| w^{\beta-1}) (\nabla w \cdot A + d_1 |w|^p) dx \leq \\
& p \int_{B_{3r}} w^\beta \eta^{p-1} e^{-b_0 w} \nabla \eta \cdot A dx - \int_{B_{3r}} \eta^p w^\beta e^{-b_0 w} B dx + \\
& \int_{B_{3r}} e^{-b_0 w} \eta^p (b_0 w^\beta + |\beta| w^{\beta-1}) d_1 |w|^p dx \leq \\
& p \int_{B_{3r}} w^\beta \eta^{p-1} e^{-b_0 w} \nabla \eta \cdot A dx + \\
& \int_{B_{3r}} \eta^p w^\beta e^{-b_0 w} (b_0 |\nabla w|^p \omega + b_1 |\nabla w|^{p-1} + d_1 |w|^{p-1}) dx + \\
& \int_{B_{3r}} e^{-\beta b_0 w} \eta^p (b_0 w^\beta + |\beta| w^{\beta-1}) d_1 |w|^p dx .
\end{aligned}$$

By Young inequality and boundedness of w in B_{3r} we obtain

$$\begin{aligned}
|\beta| \int_{B_{3r}} \eta^p w^{\beta-1} |\nabla w|^p \omega dx & \leq cp \int_{B_{3r}} w^\beta \eta^{p-1} \nabla \eta \cdot A dx + \\
& c \int_{B_{3r}} \eta^p w^\beta (b_1 |\nabla w|^{p-1} + d_1 |w|^{p-1}) dx + \\
& c \int_{B_{3r}} \eta^p (b_0 w^\beta + |\beta| w^{\beta-1}) d_1 |w|^p dx \leq \\
& c \int_{B_{3r}} \left\{ p w^\beta \eta^{p-1} |\nabla \eta| (a \omega |\nabla w|^{p-1} + b_2 |w|^{p-1}) + \right. \\
& \quad \eta^p w^\beta b_1 |\nabla w|^{p-1} + \eta^p w^{\beta+p-1} d_1 + \\
& \quad \left. + \eta^p b_0 w^{\beta+p} d_1 + |\beta| \eta^p w^{\beta+p-1} d_1 \right\} dx \leq \\
& c \int_{B_{3r}} \left\{ p w^\beta \eta^{p-1} |\nabla \eta| a \omega |\nabla w|^{p-1} + p w^{\beta+p-1} \eta^{p-1} |\nabla \eta| b_2 + \right. \\
& \quad \left. \eta^p w^\beta b_1 |\nabla w|^{p-1} + (1 + |\beta|) \eta^p w^{\beta+p-1} d_1 + \eta^p b_0 w^{\beta+p} d_1 \right\} dx .
\end{aligned}$$

Then,

$$\begin{aligned}
|\beta| \int_{B_{3r}} \eta^p w^{\beta-1} |\nabla w|^p \omega dx &\leq \\
&\leq c(b_0, M, p) \int_{B_{3r}} \left\{ w^\beta \eta^{p-1} |\nabla \eta| a |\nabla w|^{p-1} \omega dx + \right. \\
&\quad + \epsilon \eta^p w^{\beta-1} |\nabla w|^p \omega + c(\epsilon) \eta^p \frac{b_1^p}{\omega^{p-1}} w^{\beta+p-1} + \\
&\quad + \eta^{p-1} |\nabla \eta| w^{\beta+p-1} b_2 + (1 + |\beta|) \eta^p w^{\beta+p-1} d_1 \Big\} dx \leq \\
&\leq c(b_0, M, a, p) \int_{B_{3r}} \left\{ w^\beta \eta^{p-1} |\nabla \eta| |\nabla w|^{p-1} \omega dx + \right. \\
&\quad + \epsilon \eta^p w^{\beta-1} |\nabla w|^p \omega + c(\epsilon) \eta^p \frac{b_1^p}{\omega^{p-1}} w^{\beta+p-1} + \\
&\quad + w^{\beta+p-1} |\nabla \eta|^p \omega + \eta^p w^{\beta+p-1} \frac{b_2^{\frac{p}{p-1}}}{\omega^{\frac{1}{p-1}}} + \\
&\quad \left. + (1 + |\beta|) \eta^p w^{\beta+p-1} d_1 \right\} dx.
\end{aligned}$$

We set $V = \frac{b_2^{\frac{p}{p-1}}}{\omega^{\frac{1}{p-1}}} + \frac{b_1^p}{\omega^{p-1}} + d_1$ in order to get short the previous inequality.

We obtain

$$\begin{aligned}
(18) \quad \int_{B_{3r}} \eta^p w^{\beta-1} |\nabla w|^p \omega dx &\leq \\
&\leq c(1 + |\beta|^{-1})^p \int_{B_{3r}} \left\{ |\nabla \eta|^p w^{\beta+p-1} \omega + V \eta^p w^{\beta+p-1} \right\} dx.
\end{aligned}$$

Now the proof follows the lines of Theorem 4.3 in [5]. We set

$$\mathcal{U}(x) = \begin{cases} w^q(x) & \text{where } pq = p + \beta - 1 \quad \text{if } \beta \neq 1 - p \\ \log w(x) & \text{if } \beta = 1 - p \end{cases}$$

by (18) we have

$$\begin{aligned}
(19) \quad \int_{B_{3r}} \eta^p |\nabla \mathcal{U}|^p \omega(x) dx &\leq c|q|^p (1 + |\beta|^{-1})^p \left\{ \int_{B_{3r}} |\nabla \eta|^p \mathcal{U}^p \omega(x) dx + \right. \\
&\quad \left. + \int_{B_{3r}} V \eta^p \mathcal{U}^p dx \right\}, \beta \neq 1 - p
\end{aligned}$$

while

$$(20) \quad \int_{B_{3r}} \eta^p |\nabla \mathcal{U}|^p \omega(x) dx \leq c \left\{ \int_{B_{3r}} |\nabla \eta|^p \omega(x) dx + \int_{B_{3r}} V \eta^p dx \right\}$$

if $\beta = 1 - p$. Let us start with the case $\beta = 1 - p$. By Theorem 2.3 we have

$$\int_{B_{3r}} V \eta^p dx \leq c \phi \left(\frac{V}{\omega}; \text{diam } \Omega \right) \int_{B_{3r}} |\nabla \eta|^p \omega(x) dx ,$$

and from (20)

$$\int_{B_{3r}} \eta^p |\nabla \mathcal{U}|^p \omega(x) dx \leq c \int_{B_{3r}} |\nabla \eta|^p \omega(x) dx .$$

Let B_h be a ball contained in B_{2r} . Choosing $\eta(x)$ so that $\eta(x) = 1$ in B_h , $0 \leq \eta \leq 1$ in $B_{3r} \setminus B_h$ and $|\nabla \eta| \leq \frac{3}{h}$, we get

$$\|\nabla \mathcal{U}\|_{L_v^p(B_h)} \leq c \frac{\omega(B_h)^{\frac{1}{p}}}{h} .$$

By Poincaré inequality (6) and John–Nirenberg lemma (see [1]) we have $\mathcal{U}(x) = \log w(x) \in BMO_v$. Then there exist two positive constants p_0 and c , such that

$$(21) \quad \left(\int_{B_{2r}} e^{p_0 \mathcal{U}} \omega(x) dx \right)^{\frac{1}{p_0}} \left(\int_{B_{2r}} e^{-p_0 \mathcal{U}} \omega(x) dx \right)^{\frac{1}{p_0}} \leq c .$$

Let us consider the following family of seminorms

$$\Phi(p, h) = \left(\int_{B_h} |w|^p \omega(x) dx \right)^{1/p}, \quad p \neq 0 .$$

By (21) we have

$$\frac{1}{\omega(B_{2r})^{1/p_0}} \Phi(p_0, 2r) \leq c \omega(B_{2r})^{1/p_0} \Phi(-p_0, 2r) .$$

In the case (19) by Corollary 2.1 we obtain

$$(22) \quad \int_{B_{3r}} |\nabla \mathcal{U}|^p \eta^p \omega(x) dx \leq c \left\{ (|q|^p + 1) \left(1 + \frac{1}{|\beta|} \right)^p \int_{B_{3r}} |\nabla \eta|^p \mathcal{U}^p \omega(x) dx + \right. \\ \left. + \left[\frac{1}{\phi^{-1} \left(\frac{V}{\omega}; |q|^{-p} \left(1 + \frac{1}{|\beta|} \right)^{-p} \right)} \right]^{n+p} \int_{B_{3r}} \eta^p \mathcal{U}^p \omega(x) dx \right\} .$$

By Sobolev inequality we have

$$(23) \quad \left(\int_{B_{3r}} |\eta \mathcal{U}|^{kp} \omega(x) dx \right)^{\frac{1}{k}} \leq c \omega(B)^{\frac{1}{k}-1} \left\{ (|q|^p + 2) \left(1 + \frac{1}{|\beta|} \right)^p \right. \\ \left. + \int_{B_{3r}} |\nabla \eta|^p \mathcal{U}^p \omega(x) dx + \left[\frac{1}{\phi^{-1} \left(\frac{V}{\omega}; |q|^{-p} \left(1 + \frac{1}{|\beta|} \right)^{-p} \right)} \right]^{n+p} \int_{B_{3r}} \eta^p \mathcal{U}^p \omega(x) dx \right\}$$

where c is a positive constant independent of ω .

Now we choose the function η . Let r_1 and r_2 be real numbers such that $r \leq r_1 < r_2 \leq 2r$ and let the function η be chosen so that $\eta(x) = 1$ in B_{r_1} , $0 \leq \eta(x) \leq 1$ in B_{r_2} , $\eta(x) = 0$ outside B_{r_2} , $|\nabla \eta| \leq \frac{c}{r_2 - r_1}$ for some fixed constant c . We have

$$\left(\int_{B_{r_1}} \mathcal{U}^{kp} \omega(x) dx \right)^{\frac{1}{k}} \leq c \omega(B)^{\frac{1}{k}-1} \frac{1}{(r_2 - r_1)^p} (|q|^p + 2) \cdot \\ \cdot \left(1 + \frac{1}{|\beta|} \right)^p \left[\frac{1}{\phi^{-1} \left(\frac{V}{\omega}; |q|^{-p} \left(1 + \frac{1}{|\beta|} \right)^{-p} \right)} \right]^{n+p} \int_{B_{r_2}} \mathcal{U}^p \omega(x) dx.$$

Setting $\gamma = pq = p + \beta - 1$ and recalling that $\mathcal{U}(x) = w^q(x)$, we get

$$(24) \quad \Phi(k\gamma, r_1) \geq c^{\frac{1}{\gamma}} \omega(B)^{\frac{1}{\gamma}(\frac{1}{k}-1)} (|q|^p + 2)^{\frac{1}{\gamma}} \cdot \\ \cdot \left[\frac{1}{\phi^{-1} \left(\frac{V}{\omega}; |q|^{-p} \right)} \right]^{\frac{n+p}{\gamma}} \frac{1}{(r_2 - r_1)^{\frac{1}{p}}} \Phi(\gamma, r_2),$$

for negative γ . This is the inequality we are going to iterate. If $\gamma_i = k^i p_0$ and $r_i = r + \frac{r}{2^i}$, $i = 1, 2, \dots$ iteration of (24) and use of Lemma 2.1 yield

$$\Phi(-\infty, r) \geq c(p, a, \phi_{\frac{V}{\omega}}, \text{diam } \Omega) \omega(B_r)^{\frac{1}{p_0}} \Phi(-p_0, 2r).$$

Therefore by Hölder inequality,

$$\Phi(p'_0, 2r) \leq \Phi(p_0, 2r) \omega(B_r)^{\frac{1}{p'_0} - \frac{1}{p_0}}, \quad p'_0 \leq p_0.$$

So we obtain

$$\omega^{-1}(B_{2r}) \Phi(1, 2r) \leq c \Phi(-\infty, r)$$

where $c \equiv c(p, a, \phi_{\frac{v}{\omega}}, \text{diam } \Omega)$ and the result follows. \square

The next result is a weak Harnack inequality for weak subsolutions. The proof is essentially the same of the proof of the previous one.

Theorem 3.2. *Let u be a non negative weak subsolution of equation (13) in Ω satisfying (15) and (16). Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \leq M$ in B_{3r} .*

$$\max_{B_r} u \leq c \left\{ \omega^{-1}(B_{2r}) \int_{B_{2r}} u \omega dx + h(r) \right\}$$

$$\text{where } h(r) = \left[\phi \left(\left(\frac{e}{\omega} \right)^{\frac{p}{p-1}}; 3r \right) + \phi \left(\frac{g}{\omega}; 3r \right) \right]^{\frac{1}{p}} + \left[\phi \left(\frac{f}{\omega}; 3r \right) \right]^{\frac{1}{p-1}}.$$

If we take a non negative weak solution, we can put together the two previous results.

Theorem 3.3. *Let u be a non negative weak solution of equation (13) in Ω satisfying (15) and (16). Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \leq M$ in B_{3r} . Then there exists c depending on n, M, a_0, b_0, p and the weight v such that*

$$\max_{B_r} u \leq c \left\{ \min_{B_r} u + h(r) \right\}$$

$$\text{where } h(r) = \left[\phi \left(\left(\frac{e}{\omega} \right)^{\frac{p}{p-1}}; 3r \right) + \phi \left(\frac{g}{\omega}; 3r \right) \right]^{\frac{1}{p}} + \left[\phi \left(\frac{f}{\omega}; 3r \right) \right]^{\frac{1}{p-1}}.$$

Now, as a simple consequence of Harnack inequality, we get some regularity results for weak solutions of (13). The proof is an immediate consequence of Harnack inequality so we omit it.

Theorem 3.4. *Let u be a weak solution of equation (13) in Ω satisfying (15) and (16). Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \leq M$ in B_{3r} . Then u is continuous in Ω .*

If we assume more restrictive assumptions on the lower order terms we obtain the following refinement of the previous one.

Theorem 3.5. *Let u be a weak solution of equation (13) in Ω satisfying (15) and*

$$a, b_0 \in \mathbb{R}, \left(\frac{b}{\omega} \right)^{\frac{p}{p-1}}, \left(\frac{b_1}{\omega} \right)^p, \frac{d}{\omega}, \left(\frac{e}{\omega} \right)^{\frac{p}{p-1}}, \frac{f}{\omega}, \frac{g}{\omega} \in L_v^{1,p-\varepsilon}(\Omega), \quad \varepsilon > 0.$$

Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \leq M$ in B_{3r} . Then u is locally Hölder continuous in Ω .

4. APPLICATION TO NON VARIATIONAL DEGENERATE EQUATIONS

As an application of the results in the previous section, we prove continuity and Hölder continuity estimates for the gradient of solutions of some quasilinear non variational elliptic equations. The equations we are going to consider are degenerate elliptic with respect to a power of a strong A_∞ weight.

Let Ω be a bounded domain in \mathbb{R}^n ($n \geq 3$) and v a strong A_∞ weight in \mathbb{R}^n . We consider the equation

$$(25) \quad Qu = a^{ij}(x, u, \nabla u) u_{x_i x_j} + b(x, u, \nabla u) = 0, \quad \text{in } \Omega.$$

We assume the functions $a^{ij}(x, u, p)$, $b(x, u, p)$ to be differentiable in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and the following degenerate ellipticity condition

$$(26) \quad \exists \lambda > 0 : \lambda^{-1} \omega(x) |\xi|^2 \leq a^{ij}(x, u, p) \xi_i \xi_j \leq \lambda \omega(x) |\xi|^2$$

for a.e. x in Ω , $\forall u \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $\forall \xi \in \mathbb{R}^n$ where $\omega = v^{1-\frac{2}{n}}$.

Definition 4.1. Let u be a $H_{v, \text{loc}}^{2,2}(\Omega)$ function. We say that u is a $H_{v, \text{loc}}^{2,2}(\Omega)$ solution if there exists a ball B in Ω such that u satisfies (25) almost everywhere in B .

Now we prove the following

Theorem 4.1. Let Ω be a bounded domain in \mathbb{R}^n and let u be a $H_{v, \text{loc}}^{2,2}(\Omega)$ solution of equation (25) satisfying (26). We set

$$f(x) = \sup\{|a_u^{ij}(x, u, \nabla u)|, |a_x^{ij}(x, u, \nabla u)|, |b(x, u, \nabla u)|\}$$

and assume that $\left(\frac{f}{\omega}\right)^2 \in S'_v(\Omega)$ and for any ball B_r in Ω there exist positive constants M and K such that $|\nabla u| \leq M$ and $\frac{|a_p^{ij}(x, u(x), \nabla u(x))|}{\omega(x)} \leq K$ in B_r .

Then there exists $0 < \sigma \leq 1$ depending on λ , n , M , K and the weight v such that for any $0 < \rho < r$

$$\text{osc}_{B_\rho} u_{x_l} \leq c \left[\phi \left(\frac{f^2}{\omega^2}, 3\rho \right) \right]^{\sigma/2}, \quad l = 1, 2, \dots, n$$

where ϕ is the function in the Definition 2.3.

Proof. For $k = 1, 2, \dots, n$ we have

$$(27) \quad \int_{B_r} (a^{ij}(x, u, \nabla u) u_{x_i x_j} + b(x, u, \nabla u)) \varphi_{x_k} = 0 \quad \forall \varphi \in H_{0,v}^{1,2}(B_r).$$

There is no loss of generality in assuming $u \in C^3(\Omega)$. This assumption can be removed later via a density argument. Since

$$(28) \quad \int_{B_r} a^{ij}(x, u, \nabla u) u_{x_i x_j} \varphi_{x_k} dx = \int_{B_r} (-a_{x_k}^{ij} u_{x_i x_j} \varphi - a_u^{ij} u_{x_k} u_{x_i x_j} \varphi \\ - a_{p_j}^{im} u_{x_j x_k} u_{x_i x_m} \varphi - a^{ij} u_{x_i x_j x_k} \varphi) dx$$

and

$$(29) \quad - \int_{B_r} a^{ij} u_{x_i x_j x_k} \varphi dx = \int_{B_r} (a^{ij} u_{x_j x_k} \varphi_{x_i} + a_{x_i}^{ij} u_{x_j x_k} \varphi \\ + a_u^{ij} u_{x_i} u_{x_j x_k} \varphi + a_{p_m}^{ij} u_{x_m x_i} u_{x_j x_k} \varphi) dx$$

then (27) reads

$$(30) \quad \int_{B_r} \{a^{ij} u_{x_k x_j} \varphi_{x_i} + (a_m^{ij} u_{x_m x_i} u_{x_j x_k} + a^j u_{x_k x_j} + b_k^{ij} u_{x_i x_j}) \varphi + b \varphi_{x_k}\} dx = 0$$

where

$$a_m^{ij} = a_{p_m}^{ij}(x, u, \nabla u) - a_{p_j}^{im}(x, u, \nabla u) \\ a^j = a_u^{ij}(x, u, \nabla u) u_{x_i} + a_{x_i}^{ij}(x, u, \nabla u) \\ b_k^{ij} = -a_u^{ij}(x, u, \nabla u) u_{x_k} - a_{x_k}^{ij}(x, u, \nabla u).$$

We choose $\varphi = u_{x_k} \eta(x)$ as test function in (30), where $\eta \geq 0$, $\eta \in C_0^1(B_r)$ so we get

$$(31) \quad \int_{B_r} \{a^{ij} u_{x_k x_i} u_{x_k x_j} \eta + \frac{1}{2} a^{ij} v_{x_j} \eta_{x_i} + \frac{1}{2} a_m^{ij} u_{x_m x_i} v_{x_j} \eta \\ + \frac{1}{2} a^j v_{x_j} \eta + b_k^{ij} u_{x_i x_j} u_{x_k} \eta + b \Delta u \eta + b u_{x_k} \eta_{x_k}\} dx = 0$$

where we set $v = |\nabla u|^2$.

Let $\gamma > 0$ and set $w = w_l^+ = \gamma u_{x_l} + v$, $l = 1, \dots, n$. From (31) and (30) we obtain

$$\int_{B_r} \{a^{ij} u_{x_i x_k} u_{x_k x_j} \eta + (\frac{1}{2} a^{ij} w_{x_j} + b u_{x_i} + \frac{1}{2} \gamma b \delta_i^l) \eta_{x_i}\} dx = \\ - \int_{B_r} \{\frac{1}{2} a_m^{ij} u_{x_m x_i} w_{x_j} + \frac{1}{2} a^j w_{x_j} + (\frac{1}{2} \gamma b_l^{ij} + b_k^{ij} u_{x_k} + b \delta_i^j) u_{x_i x_j}\} \eta dx$$

and then

$$\begin{aligned}
(32) \quad & \int_{B_r} (a^{ij}w_{x_j} + 2bu_{x_i} + \gamma b\delta_i^l)\eta_{x_i} dx \leq \\
& \leq \int_{B_r} \left\{ -2\lambda^{-1}\omega(x)|D^2u|^2 - a_m^{ij}u_{x_mx_i}w_{x_j} - a^j w_{x_j} - \right. \\
& \quad \left. - (\gamma b_{ij}^l + 2b_{ij}^k u_{x_k} + 2b\delta_i^j)u_{x_ix_j} \right\} \eta dx \leq \\
& \leq \int_{B_r} \left\{ -2\lambda^{-1}\omega(x)|D^2u|^2 + \left(\sum_{m,i} (a_m^{ij}w_{x_j})^2 \right)^{\frac{1}{2}} |D^2u| + \right. \\
& \quad \left. + |\nabla w|f(x) + |D^2u|f(x) \right\} \eta dx \leq \\
& \leq \int_{B_r} \left\{ -2\lambda^{-1}\omega(x)|D^2u|^2 + \lambda^{-1}\omega(x)|D^2u|^2 + \frac{\lambda}{\omega(x)} \sum_{m,i} (a_m^{ij}w_{x_j})^2 + \right. \\
& \quad \left. + \lambda^{-1}\omega(x)|\nabla w|^2 + \frac{2\lambda f^2}{\omega(x)} + \lambda^{-1}\omega(x)|D^2u|^2 \right\} \eta dx \leq \\
& \leq c(K, \lambda) \int_{B_r} \left\{ \omega(x)|\nabla w|^2 + \frac{f^2}{\omega(x)} \right\} \eta dx.
\end{aligned}$$

The previous inequality shows that $w(x) = w_l^+(x)$ is a local weak sub-solution of the equation

$$(33) \quad -(\tilde{a}_{ij}w_{x_i})_{x_j} - c(K, \lambda)\omega|\nabla w|^2 = \frac{f^2}{\omega} - (F_i(x))_{x_i}$$

where $\tilde{a}_{ij}(x) = a^{ij}(x, u(x), \nabla u(x))$ and

$$F_i(x) = -2b(x, u(x), \nabla u(x))u_{x_i}(x) - \gamma b(x, u(x), \nabla u(x))\delta_i^l.$$

We note that $|F_i| \leq c(M)f$, $i = 1, 2, \dots, n$. This implies $\left(\frac{F_i}{\omega}\right)^2 \in S'_v(B_r)$ and

$$\left\| \left(\frac{F_i}{\omega} \right)^2 \right\|_{S'_v(B_r)} \leq \left\| \left(\frac{f}{\omega} \right)^2 \right\|_{S'_v(B_r)}.$$

Now fix $0 < \rho < \min\{1, \frac{1}{3}r\}$ and choose $1 \leq h \leq n$ such that

$$\text{osc}_{B_{3\rho}} u_{x_h} \geq \text{osc}_{B_{3\rho}} u_{x_l} \quad \forall l = 1, 2, \dots, n.$$

Now we fix γ a sufficiently large positive number. It turns out that a convenient choice is $\gamma = 10nM$. Then we have

$$\begin{aligned} \operatorname{osc}_{B_{3\rho}} w_h^+ &\leq \operatorname{osc}_{B_{3\rho}}(10nMu_{x_h}) + \operatorname{osc}_{B_{3\rho}} |\nabla u|^2 \leq \\ &\leq 10nM \operatorname{osc}_{B_{3\rho}} u_{x_h} + \operatorname{osc}_{B_{3\rho}} \left(\sum_{i=1}^n u_{x_i}^2 \right) \leq \\ &\leq 10nM \operatorname{osc}_{B_{3\rho}} u_{x_h} + 2M \operatorname{osc}_{B_{3\rho}} \left(\sum_{i=1}^n u_{x_i} \right) \leq 12nM \operatorname{osc}_{B_{3\rho}} u_{x_h} \end{aligned}$$

and

$$\operatorname{osc}_{B_{3\rho}} w_h^+ \geq 10nM \operatorname{osc}_{B_{3\rho}} u_{x_h} - \operatorname{osc}_{B_{3\rho}} \left(\sum_{i=1}^n u_{x_i}^2 \right) \geq 8nM \operatorname{osc}_{B_{3\rho}} u_{x_h}.$$

Putting together the previous inequalities we obtain

$$(34) \quad 8nM \operatorname{osc}_{B_{3\rho}} u_{x_h} \leq \operatorname{osc}_{B_{3\rho}} w_h^+ \leq 12nM \operatorname{osc}_{B_{3\rho}} u_{x_h}.$$

The same argument applies to the function $w_h^- = -\gamma u_{x_h} + v$. Arguing in the same way we get

$$(35) \quad 8nM \operatorname{osc}_{B_{3\rho}} u_{x_h} \leq \operatorname{osc}_{B_{3\rho}} w_h^- \leq 12nM \operatorname{osc}_{B_{3\rho}} u_{x_h}.$$

The functions $\sup_{B_{3\rho}} w_h^+ - w_h^+$ and $\sup_{B_{3\rho}} w_h^- - w_h^-$ are supersolutions of (33) which is non linear because of the quadratic term in the gradient. However, we may apply the results in the previous section taking $p = 2$. Then, from Theorem 3.1 we get

$$(36) \quad \omega^{-1}(B_{2\rho}) \int_{B_{2\rho}} (\sup_{B_{3\rho}} w_h^+ - w_h^+) \omega dx \leq c(\sup_{B_{3\rho}} w_h^+ - \sup_{B_\rho} w_h^+ + h(\rho))$$

and

$$(37) \quad \omega^{-1}(B_{2\rho}) \int_{B_{2\rho}} (\sup_{B_{3\rho}} w_h^- - w_h^-) \omega dx \leq c(\sup_{B_{3\rho}} w_h^- - \sup_{B_\rho} w_h^- + h(\rho))$$

$$\text{where } h(\rho) = \left[\phi \left(\frac{f^2}{\omega^2}, 3\rho \right) \right]^{1/2}.$$

As a consequence of (34) we have

$$\begin{aligned}
 (38) \quad & \sup_{B_{3\rho}} w_h^+ - w_h^+ + \sup_{B_{3\rho}} w_h^- - w_h^- = \sup_{B_{3\rho}} w_h^+ + \sup_{B_{3\rho}} w_h^- - 2v \geq \\
 & \geq \sup_{B_{3\rho}} w_h^+ + \sup_{B_{3\rho}} w_h^- - 2 \sup_{B_{3\rho}} v \geq \\
 & \geq \sup_{B_{3\rho}} (10nMu_{x_h}) - \inf_{B_{3\rho}} (10nMu_{x_h}) + 2 \inf_{B_{3\rho}} v - 2 \sup_{B_{3\rho}} v \geq \\
 & \geq 10nM \operatorname{osc}_{B_{3\rho}} u_{x_h} - 4nM \operatorname{osc}_{B_{3\rho}} u_{x_h} \geq \frac{1}{2} \operatorname{osc}_{B_{3\rho}} w_h^+ \quad \forall x \in B_{3\rho}
 \end{aligned}$$

and in the same way we see that

$$(39) \quad \sup_{B_{3\rho}} w_h^+ - w_h^+ + \sup_{B_{3\rho}} w_h^- - w_h^- \geq \frac{1}{2} \operatorname{osc}_{B_{3\rho}} w_h^- \quad \forall x \in B_{3\rho}.$$

By (38) and (39), the following inequalities

$$\begin{cases} \frac{1}{4} \operatorname{osc}_{B_{3\rho}} w_h^+ > \omega^{-1}(B_{2\rho}) \int_{B_{2\rho}} (\sup_{B_{3\rho}} w_h^+ - w_h^+) \omega(x) dx \\ \frac{1}{4} \operatorname{osc}_{B_{3\rho}} w_h^- > \omega^{-1}(B_{2\rho}) \int_{B_{2\rho}} (\sup_{B_{3\rho}} w_h^- - w_h^-) \omega(x) dx \end{cases}$$

cannot be both true at the same time. Let us suppose that

$$\frac{1}{4} \operatorname{osc}_{B_{3\rho}} w_h^+ \leq \omega^{-1}(B_{2\rho}) \int_{B_{2\rho}} (\sup_{B_{3\rho}} w_h^+ - w_h^+) \omega(x) dx.$$

By (36) we are able to give an estimate for the oscillation of w_h^+ , i.e.

$$\begin{aligned}
 \operatorname{osc}_{B_{3\rho}} w_h^+ & \leq c(\sup_{B_{3\rho}} w_h^+ - \sup_{B_\rho} w_h^+ + h(\rho)) \leq \\
 & \leq c(\operatorname{osc}_{B_{3\rho}} w_h^+ - \operatorname{osc}_{B_\rho} w_h^+ + h(\rho))
 \end{aligned}$$

from which

$$\operatorname{osc}_{B_\rho} w_h^+ \leq (1 - 1/c) \operatorname{osc}_{B_{3\rho}} w_h^+ + h(\rho).$$

Now we can apply Lemma 2.2. There exist positive constants $\sigma \leq 1$, $\bar{\rho}$ and k such that

$$\operatorname{osc}_{B_\rho} w_h^+ \leq kh^\sigma(\rho) \quad \forall \rho < \bar{\rho}.$$

Then we get

$$\operatorname{osc}_{B_\rho} u_{x_l} \leq ckh^\sigma(\rho) \quad \forall \rho < \bar{\rho} \quad \forall l = 1, \dots, n$$

and the proof is complete. \square

Refining our assumptions - as a consequence of Proposition 2.1 - we get

Theorem 4.2. *Let Ω be a bounded domain in \mathbb{R}^n and u be a $H_{v,\text{loc}}^{2,2}(\Omega)$ solution of equation (25) satisfying (26). We set*

$$f(x) = \sup\{|a_u^{ij}(x, u, \nabla u)|, |a_x^{ij}(x, u, \nabla u)|, |b(x, u, \nabla u)|\}.$$

Assume that $\left(\frac{f}{\omega}\right)^2 \in L_v^{1,p-\varepsilon}(\Omega)$, for $0 < \varepsilon < p$, and for any ball B_r in Ω there exist positive constants M and K such that $|\nabla u| \leq M$ and $\frac{|a_p^{ij}(x, u(x), \nabla u(x))|}{\omega(x)} \leq K$ in B_r .

Then there exists $0 < \alpha < 1$ depending on λ, n, M, K and the weight v such that for any $0 < \rho < r$

$$\text{osc}_{B_\rho} u_{x_l} \leq c\rho^\alpha, \quad l = 1, 2, \dots, n.$$

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